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On Some New Operations on Orthomodular Lattices[†]

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Kotas conditionals are used to define six pairs of disjunction- and conjunction-like operations on orthomodular lattices. Although five of them necessarily differ from the lattice operations on elements that are not compatible, they coincide with the lattice operations on all compatible elements of the lattice and they define on the underlying set a partial order relation that coincides with the original one. Some of the new operations are noncommutative on noncompatible elements, but this does not exclude the possibility to endow them with a physical interpretation. The new operations are in general nonassociative, but for some of them a Foulis–Holland-type theorem concerning associativity instead of distributivity holds. The obtained results suggest that these new operations can serve as alternative algebraic models for the logical operations of disjunction and conjunction.

1. INTRODUCTION

Garrett Birkhoff and John von Neumann, the founding fathers of quantum logic theory, were not very satisfied with their own proposal of unrestricted treating of lattice operations (*meet* and *join*) as algebraic models of logical operations of conjunction and disjunction of experimentally testable propositions about quantum objects. They were aware of the problems that are bound to emerge when considered propositions are not compatible and they wrote in their historic 1936 paper (Birkhoff and von Neumann, 1936):

> It is worth remarking that in classical mechanics, one can easily define the meet or join of any two experimental propositions as an experimental proposition—

[†]This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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> simply by having independent observers read off the measurements which either proposition involves, and combining them logically. This is true in quantum mechanics only exceptionally—only when all the measurements involved commute (are compatible).

On the other hand, Birkhoff and von Neumann's choice of representing conjunctions and disjunctions by meets and joins is natural and obvious as far as compatible propositions are concerned: If we construct a Lindenbaum–Tarski algebra of a theory that is governed by the laws of classical logic, then meets and joins in this Boolean algebra really do represent, respectively, conjunctions and disjunctions of propositions. On noncompatible propositions, however, the interpretational problems remain. We offer a solution for this by defining six pairs ($\flat_i, \sharp_i$) of operations ($i = 1, \ldots, 6$) on an orthomodular lattice (OML) which can be interpreted as new models of conjunction and disjunction of two propositions. They are introduced using Hardegree's papers on physical conditionals on an OML (Hardegree, 1974, 1979, 1981) in which a set of generalized conditionals is discussed. By modifying these generalized conditionals, we define new pairs of disjunction-like ($a \sharp_i b$) and conjunction-like ($a \flat_i b$) operations. Actually, one new pair of the connectives found with this procedure was first introduced in (Pykacz, n.d.). Although they were introduced in another way, i.e., by making use of formal Mackey (1963) decompositions, these connectives combined with the Hardegree's results inspired the introduction of the operations put forward in this paper.

Many of the properties of pairs of disjunction- and conjunction-like operations are the same as properties of join and meet; moreover, $a \sharp_i b$ and $a \flat_i b$ coincide, respectively, with join $a \vee b$ and meet $a \wedge b$ whenever a and b are compatible. They also define on the underlying set a partial order relation which coincides with the original one. Therefore, it is possible to treat $a \sharp_i b$ and $a \flat_i b$ as new algebraic models of disjunction and conjunction of the propositions represented by a and b. We give some arguments that such an interpretation of these operations is plausible in spite of the fact that some properties of $\sharp_i$ and $\flat_i$ are rather counterintuitive. However, since on compatible propositions the new operations coincide with the traditional ones and they give rise to the same partial order structure, we argue that from the operational point of view all connectives can be treated on the same level as the traditional ones.

In order to keep the size of this paper within reasonable limits, we quote only the most important results for the purpose of this paper; more extensive results will be published elsewhere (D'Hooghe, n.d., Pykacz, n.d.). The reader interested in all the details, especially, the proofs of all quoted theorems, is referred to these papers. It should be also stressed that some of the operations considered in the present paper already appeared, in various contexts, in general different from ours, in the literature and some of their properties quoted in the present paper were already studied. This, in particular, concerns Kröger's results in the context of Boolean skew lattices quoted in Beran's (1985) book, Sections VII 6 and VII 7, with respect to our Theorems 4, 5, and 7 (for $\sharp_2$ and $\flat_2$), Roman and Rumbos' (1991) results concerning our Theorems 4 and 5 for the operation $\flat_2$, or results of Dorfer *et al.* (1996) and Länger (1998) concerning the operation $\flat_3$ in the contexts of our Theorems 5, 6, and 7. For the detailed references and discussion concerning these facts see our already mentioned papers. The definitions of all relevant notions concerning orthomodular lattices can be found in any of numerous textbooks on quantum logic, for example, Beltrametti and Cassinelli (1981), Kalmbach (1983), or Beran (1985).

2. CONNECTIVES DERIVED FROM KOTAS CONDITIONALS

2.1. Kotas Conditionals on Orthomodular Lattices

Let $\mathcal{L}$ be an orthomodular lattice (OML), with the least element **0** and greatest element **1**. If $\mathcal{L}$ represents the lattice of properties of a classical system, it is Boolean and the lattice operations of orthocomplement, meet, and join of element(s) of the lattice represent, respectively, the negation, conjunction, and disjunction of propositions that express these properties. However, in classical logic there exists yet a fourth operation on a pair of elements which does not have an unambiguous analog in the theory of OML. This operation is the implication (conditional, horseshoe) denoted $a \supset b$. To make the distinction between the classical conditional $\supset$, 'if a, then b', and the lattice operation we will write the latter as $a \to b$. If we represent two propositions of a classical logic by elements a and b of the distributive complemented lattice (Boolean algebra) $\mathcal{L}$, their disjunction is represented by the join $a \vee b$ of these two elements. In classical logic the sentence 'if a, then b' is the same as saying '(not a) or b'. In lattice-theoretic terms this means that the classical conditional is represented by $a \to b = a' \vee b$. It is important to stress that the conditional $a \to b$ is an element of the lattice, associated with the couple (a, b) via the formula $a \to b = a' \vee b$, and that it is not expressing any truth value of the implication 'if a, then b'. Using the definition $a \to b = a' \vee b$ for the classical conditional, one can prove a list of properties which, alternatively, could be used as the defining properties for the classical conditional. The explicit expression $a \to b = a' \vee b$ follows then as a theorem. Although the lattice-theoretic counterpart of the conditional is unambiguously defined by such a set of properties in a Boolean algebra, in the case of a nondistributive OML, this is not the case.

We use as the defining property for a general conditional on an OML the requirement that on compatible propositions (which define a distributive subalgebra of the OML; see, e.g., Beltrametti and Cassinelli, 1981) the conditional should coincide with the conditional of a Boolean algebra: if a and b are compatible (abbreviated aCb), then $a \to b = a' \vee b$. It was shown by Kotas (1967) that there are only six polynomials $c_i(a, b)$, $i = 1$–6, on the OML which satisfy this condition, namely:

1. $c_1(a, b) \equiv a' \vee (a \wedge b)$
2. $c_2(a, b) \equiv (a' \wedge b') \vee b$
3. $c_3(a, b) \equiv (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b')$
4. $c_4(a, b) \equiv (a \wedge b) \vee (a' \wedge b) \vee ((a' \vee b) \wedge b')$
5. $c_5(a, b) \equiv (a \wedge (a' \vee b)) \vee (a' \wedge b) \vee (a' \wedge b')$
6. $c_6(a, b) \equiv a' \vee b$

and the following holds: $a \leq b \Leftrightarrow c_i(a, b) = \mathbf{1}$ for $i = 1$–5.

2.2. New Operations Derived from Kotas Conditionals

As we already mentioned, the conditional in classical logic can be defined by the disjunction of propositions (NOT a) and b, which in lattice-theoretic terms corresponds with the join of a' and b. Alternatively, the join of two elements can be defined by the classical conditional: $a \vee b \equiv a' \to b$. Analogously, we will now define disjunction-like operations on an OML using the six conditionals quoted in the previous section. The disjunction-like operation derived from the conditional c_i will be denoted by $\sharp_i$, in other words the ith disjunction-like operation is defined by $a \sharp_i b \equiv c_i(a', b)$, and we obtain:

1. $a \sharp_1 b \equiv a \vee (a' \wedge b)$
2. $a \sharp_2 b \equiv (a \wedge b') \vee b$
3. $a \sharp_3 b \equiv (a' \wedge b) \vee (a \wedge b) \vee (a \wedge b')$
4. $a \sharp_4 b \equiv (a' \wedge b) \vee (a \wedge b) \vee ((a \vee b) \wedge b')$
5. $a \sharp_5 b \equiv (a' \wedge (a \vee b)) \vee (a \wedge b) \vee (a \wedge b')$
6. $a \sharp_6 b \equiv a \vee b$

For each operation the corresponding conjunction-like operation $\flat_i$ is defined via the De Morgan law $a \flat_i b \equiv (a' \sharp_i b')'$ and we get:

1. $a \flat_1 b \equiv a \wedge (a' \vee b)$
2. $a \flat_2 b \equiv (a \vee b') \wedge b$
3. $a \flat_3 b \equiv (a' \vee b) \wedge (a \vee b) \wedge (a \vee b')$
4. $a \flat_4 b \equiv (a' \vee b) \wedge (a \vee b) \wedge ((a \wedge b) \vee b')$
5. $a \flat_5 b \equiv (a' \vee (a \wedge b)) \wedge (a \vee b) \wedge (a \vee b')$
6. $a \flat_6 b \equiv a \wedge b$

Obviously, by definition, each pair of operations $(\sharp_i, \flat_i)$ obeys De Morgan identities for $i = 1$–6. Although in a distributive logic there is no ambiguity about which conditional to use if one wants to define the disjunction using the conditional (since all six conditionals coincide with the classical conditional on compatible elements), in a nondistributive OML the six conditionals do not coincide with each other and in general they differ from the classical conditional. As a consequence, in general the new disjunction-like operations differ from each other and from the ordinary join on noncompatible elements.

It follows from the definition that operations $\sharp_3$, $\flat_3$, $\sharp_6$, and $\flat_6$ are commutative. The other operations are not commutative on all pairs of elements of an OML. However, there is a 'commutative duality' between operations $\sharp_1$ and $\sharp_2$, and operations $\sharp_4$ and $\sharp_5$ (the same holds for the respective pairs of conjunction-like operations $\flat_i$, $i = 1, 2; 4, 5$): $a \sharp_1 b = b \sharp_2 a$ and $a \sharp_4 b = b \sharp_5 a$, which again follows directly from the definition.

3. NEW OPERATIONS VERSUS COMPATIBILITY OF PROPOSITIONS

3.1. Old and New Operations versus Compatibility of Propositions

Close links between the operations $\sharp_i$, $\flat_i$ and the lattice-theoretic operations of join and meet are established by the following

Theorem 1. Let $\mathcal{L}$ be an orthomodular lattice. For any $a, b \in \mathcal{L}$ the following conditions are equivalent, for $i = 1$–5:

$$aCb \tag{1}$$

$$a \sharp_i b = a \vee b \tag{2}$$

$$a \flat_i b = a \wedge b \tag{3}$$

Since $a \leq b$ implies that aCb, and aCa', $aC\mathbf{0}$, $aC\mathbf{1}$, the following properties are an immediate consequence of the previous theorem:

Theorem 2:

1. $\sharp_i$ and $\flat_i$ are idempotent: $a \sharp_i a = a$, $a \flat_i a = a$ for $i = 1$–6.
2. $a \flat_i \mathbf{0} = \mathbf{0}$, $a \flat_i \mathbf{1} = a$, $a \sharp_i \mathbf{0} = a$, $a \sharp_i \mathbf{1} = \mathbf{1}$ for $i = 1$–6.
3. $\sharp_i$ and $\flat_i$ satisfy the law of excluded middle and the law of contradiction, for $i = 1$–6: $a \sharp_i a' = \mathbf{1}$, $a \flat_i a' = \mathbf{0}$.
4. $\sharp_i$ and $\flat_i$ satisfy the 'orthomodular identity' for $i = 1$–6: if $a \leq b$, then $b = a \sharp_i (a' \flat_i b)$.

Theorem 3. $\sharp_i$ and $\flat_i$, $i = 1$–6 are commutative on compatible propositions.

3.2. Distributivity, Commutativity, and Associativity

Since the new operations resemble in many aspects lattice operations on an OML it is not surprising that they are in general nondistributive, which can be demonstrated of course only when the considered elements do not belong to the same Boolean subalgebra of an OML. For example, let us consider $\mathcal{L} = G_{12}$ represented in Fig. 1. The nondistributivity can then be easily checked on G_{12}, for instance,

$$a \sharp_3 (c \flat_3 e) = a \sharp_3 (c \wedge e) = a \sharp_3 0 = a \vee 0 = a$$

while

$$(a \sharp_3 c) \flat_3 (a \sharp_3 e) = (a \vee c) \flat_3 0 = b' \flat_3 0 = b' \wedge 0 = 0$$

and

$$a \flat_3 (c' \sharp_3 e) = a \flat_3 (c' \vee e) = a \flat_3 c' = a \wedge c' = a$$

while

$$(a \flat_3 c') \sharp_3 (a \flat_3 e) = (a \wedge c') \sharp_3 c' = a \sharp_3 c' = a \vee c' = c'$$

Since in this example aCc, cCe, and aCc', $c'Ce$, i.e., we were *focusing*, respectively, on the elements c and c', we see that no counterpart of the

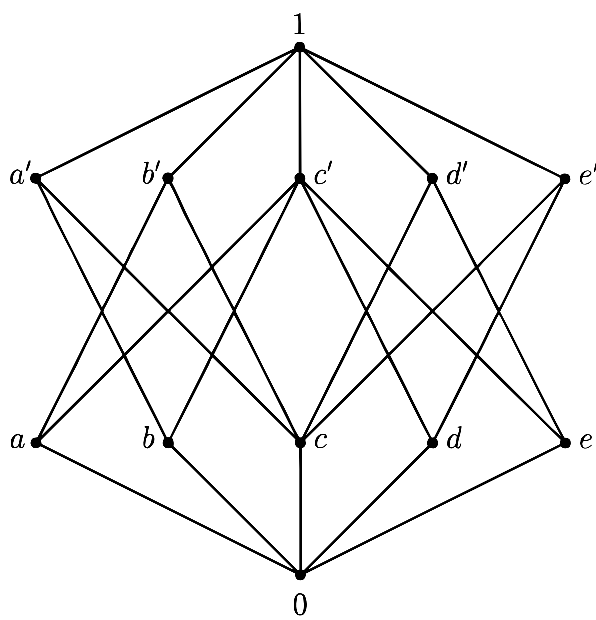

Fig. 1.

Foulis–Holland theorem (Foulis, 1962; Holland, 1963; see also Beltrametti and Cassinelli, 1981, p. 128; Kalmbach, 1983, p. 25) holds for $\#_3$ and $\flat_3$ and the same can be checked for $i = 1, 2, 4, 5$. Of course, because of Theorem 1, new operations applied only to noncompatible propositions may show features by which they differ from traditional lattice operations join ($\sharp_6$) and meet ($\flat_6$), for example nonassociativity and noncommutativity. Noncommutativity for $\sharp_i$, $\flat_i$, $i = 1, 2, 4, 5$, can also be demonstrated on G_{12}, for instance, for $i = 2$: $a \sharp_2 e = e$, while $e \sharp_2 a = a$. If a and b are compatible, operations $\flat_i$, $\sharp_i$ ($i = 1, 2, 4, 5$) are obviously commutative since in such a case they coincide, respectively, with the traditional lattice operations of meet and join. The converse statement is also true:

Theorem 4. $a \sharp_i b = b \sharp_i a$ iff aCb for $i = 1, 2, 4, 5$.

Corollary. $a \flat_i b = b \flat_i a$ iff aCb for $i = 1, 2, 4, 5$.

The general nonassociativity of operations $\sharp_i$, $\flat_i$ for $i = 1, 2, 3, 4, 5$ can be checked on G_{12} as well, for example:

$$a \sharp_3 (b \sharp_3 e) = a \sharp_3 0 = a \vee 0 = a$$

while

$$(a \sharp_3 b) \sharp_3 e = (a \vee b) \sharp_3 e = c' \sharp_3 e = c' \vee e = c'$$

and

$$a \flat_3 (b \flat_3 e) = a \flat_3 c' = a \wedge c' = a$$

while

$$(a \flat_3 b) \flat_3 e = (a \wedge b) \flat_3 e = 0 \flat_3 e = 0 \wedge e = 0$$

However, it is a surprising fact that although in general the new operations are nonassociative, a Foulis–Holland-type theorem concerning associativity instead of distributivity holds for some of them:

Theorem 5. If, in an orthomodular lattice, one of the elements a, b, c is compatible with the other two, then $\{a, b, c\}$ is an associative triple with respect to both operations $\sharp_i$ and $\flat_i$ for $i = 1, 2, 3$.

Theorem 5 facilitates attempts at interpreting $\sharp_i$ as OR and $\flat_i$ as AND for $i = 1, 2, 3$ in spite of the general lack of associativity of $\sharp_i$ and $\flat_i$, for it secures the unique meaning of propositions '*a* AND *b* AND *c*' and '*a* OR *b* OR *c*' for all possible permutations of $\{a, b, c\}$ when one of these experimental propositions is simultaneously (but possibly separately) verifiable with the remaining two, although the whole triple $\{a, b, c\}$ does not have to belong to the same Boolean subalgebra of an OML. More precisely, even if truth values of conjunctions and disjunctions of two of the pairs (a, b), (a, c), and (b, c) have to be checked in different experiments while the truth value of the conjunction and disjunction of the remaining pair of propositions cannot be experimentally verified at all because these propositions do not belong to the same Boolean subalgebra, nevertheless, the statements 'a AND b AND c' and 'a OR b OR c' remain meaningful.

We did not prove the counterpart of Theorem 5 for $i = 4, 5$. However, since we found no counterexample, we suspect that it might hold true.

3.3 The New Operations versus Compatibility of Propositions

In Theorem 1 it was shown that each pair ($\flat_i$, $\sharp_i$) of new operations coincides with the ordinary lattice-theoretic operations meet and join on and only on compatible elements. This, however, did not exclude the possibility that on noncompatible elements some new operations could coincide. This possibility is excluded by the following theorem, which is a generalization of Theorem 1:

Theorem 6. Let $\mathscr{L}$ be an orthomodular lattice. For any $a, b \in \mathscr{L}$ the following conditions are equivalent for $i \neq j$, $i, j \in \{1, \ldots, 6\}$:

$$aCb \tag{1'}$$

$$a \sharp_i b = a \sharp_j b \tag{2'}$$

$$a \flat_i b = a \flat_j b \tag{3'}$$

One might doubt whether this theorem is valid in the case of OMLs that contain very few elements. However, notice that the smallest non-Boolean OML, the so-called *Chinese lantern* or *MO2* (see, e.g., Kalmbach, 1983, p. 16) contains *exactly* six elements: $\mathbf{0} \leq a, a', b, b' \leq \mathbf{1}$ (none of a, a', b, b' being comparable). Thus we see that the number of elements in this smallest non-Boolean OML is just big enough to allow Theorem 6 to be valid and we indeed get, for example, $a \sharp_1 b = a$, $a \sharp_2 b = b$, $a \sharp_3 b = 0$, $a \sharp_4 b = b'$, $a \sharp_5 b = a'$, and $a \sharp_6 b = 1$. Therefore, each element of *MO2* represents in this way exactly one disjunction-like operation. In a similar way we can show that the same is true for the set of conjunction-like operations $\flat_i$, $i = 1$–6. This demonstrates that even in the 'minimal' non-Boolean OML *MO2*, which contains exactly the same amount of elements as there are different pairs ($\sharp_i$, $\flat_i$) of disjunction- and conjunction-like operations, the minimal number of elements of this OML does not prevent Theorem 6 from being valid.

4. MUTUAL DEFINABILITY OF OPERATIONS

The new operations $\sharp_i$ and $\flat_i$, $i = 1, \ldots, 5$, were defined with the aid of the lattice operations of join and meet, and the operation of orthocomplementation. The natural question arises whether it is possible to go in the opposite direction, i.e., to express the lattice operations of join and meet by operations $\sharp_i$ and $\flat_i$ (and, possibly, orthocomplementation). The following theorem answers this question in the positive.

Theorem 7. For any two elements a, b of an orthomodular lattice $\mathcal{L}$

$$\begin{aligned} a \vee b &= (a' \flat_1 b) \sharp_1 a \\ &= a \sharp_2 (b \flat_2 a') \\ &= (a \sharp_3 b) \sharp_3 (a \flat_3 b) \\ &= (a \flat_3 b') \sharp_3 b \\ &= (a' \flat_3 b) \sharp_3 a \\ &= (b \sharp_4 a) \sharp_4 a \\ &= a \sharp_5 (a \sharp_5 b) \\ a \wedge b &= (a' \sharp_1 b) \flat_1 a \\ &= a \flat_2 (b \sharp_2 a') \\ &= (a \sharp_3 b) \flat_3 (a \flat_3 b) \\ &= (a \sharp_3 b') \flat_3 b \\ &= (a' \sharp_3 b) \flat_3 a \\ &= (b \flat_4 a) \flat_4 a \\ &= a \flat_5 (a \flat_5 b) \end{aligned}$$

For $\sharp_6$, $\flat_6$ the answer is of course trivial, since these operations coincide with the traditional join and meet.

Of course, Theorem 7 allows one to express in many ways any of the studied operations by (any of) the other(s) and orthocomplementation. However, the following example, in which we express $\sharp_1$ by $\sharp_3$ and shows that the obtained formulas might be rather lengthy:

$$a \sharp_1 b = \{a \sharp_3 [(a \sharp_3 b)' \sharp_3 a]\}' \sharp_3 a$$

It is an open question which of such formulas (if any) could be written in a more economical way.

5. POSSIBLE PHYSICAL AND LOGICAL INTERPRETATION OF $\sharp_i$ AND $\flat_i$

5.1. Partial Order Defined by the New Operations

It can be shown (Kotas, 1967) that the five conditionals c_i, $i = 1$–5, satisfy the following condition:

$$a \leq b \Leftrightarrow c_i(a, b) = \mathbf{1}, \qquad i = 1\text{–}5$$

which implies that the partial order relation $\leq$ can be reconstructed from each of these five conditionals. Since these conditionals were used in the definition of the disjunction-like operations $\sharp_i$, $i = 1$–5, this means that the partial order relation $\leq$ can be defined via each disjunction-like operation as follows:

$$a \leq b \Leftrightarrow_{df} a' \sharp_i b = \mathbf{1}$$

This shows that there are two possibilities: (1) the partial order structure on the OML is given and defines meet, join, and, as a consequence, all disjunction- and conjunction-like operations on the OML, or (2) one of the disjunction-like or conjunction-like operations $\sharp_i$ or $\flat_i$, $i = 1$–5, is given, from which the partial order is constructed using the equation mentioned above and, consequently, all remaining operations: join, meet, and the other disjunction-like and conjunction-like operations, are defined. From the traditional join and meet operations the partial order relation can also be deduced, but in a slightly different way. In conclusion, each of the 12 operations $\sharp_i$, $\flat_i$, $i = 1, \ldots 6$, is equally good to define the partial order relation of the lattice.

5.2. New Operations as Models of Logical Disjunction and Conjunction

One of the operations studied in this paper, namely $\flat_2$, was already studied in a physical context by Roman and Rumbos (1991), who argued that it might serve as a better model for the conjunction of two propositions about a quantum system. However, Roman and Rumbos, who interpreted $\flat_2$ on the lattice of projectors as yielding the 'closest' projector associated with the composition of two, possibly noncommuting, projectors, did not reflect a lot on the possible physical advantages or disadvantages of noncommutativity of this operation. We shall argue now that although all operations considered in this paper are operationally indistinguishable, the noncommutativity of operations 1, 2, 4, and 5 opens new interpretational possibilities that do not exist for operations 3 and 6.

Theorem 1 implies that we cannot distinguish between the lattice operations and the operations $\sharp_i$ and $\flat_i$ if we have access only to compatible elements of a lattice. According to the standard interpretation of (quantum) logics, i.e., orthomodular lattices associated to (quantum) physical systems, elements of a logic represent experimentally verifiable propositions about the associated physical system. One of the crucial differences between classical physical systems (whose logics are Boolean algebras) and quantum systems is that for a quantum system there do exist propositions, represented by noncompatible elements of a logic, that cannot be verified simultaneously. However, since all comparable elements of a lattice are compatible, in order to gain knowledge about the whole order-theoretic structure of a set of propositions, it suffices to perform experiments involving only pairs of compatible propositions: simultaneous verification of noncompatible propositions is neither possible from the experimental nor necessary from the theoretical point of view. In view of these considerations and also of Theorem 1, it seems that we cannot distinguish between the new operations and the traditional lattice operations of meet and join by making *real experiments* involving pairs of propositions, so it is a matter of choice which ones among these operations are better suited to be used as algebraic models of conjunctions and disjunctions of propositions about quantum systems. Therefore, from the operational point of view all six pairs of operations are equivalent and it is meaningless to argue which pair of connectives is the ‘most natural’ one. Nevertheless although the noncommutativity of some of these operations could be considered as a mathematical disadvantage, from a physical point of view this feature makes them especially interesting. The noncommutativity of operations 1, 2, 4, and 5 could serve as a tool to absorb the impossibility of simultaneous verification of some properties of quantum systems into the logicoalgebraic formulation of quantum theory and could allow one to describe in a natural way ‘sequential’ tests in which we verify two (possibly noncompatible) properties of a quantum system not simultaneously, but one after another. In such a sense, paraphrasing George Orwell’s *Animal Farm*, we could say about the six pairs of disjunction- and conjunction-like operations on OMLs studied in this paper that from an operational point of view, “All operations are equal, but some of them are more equal than others.”

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